Chapter 4

A New Class I Intermittency

4.1 Introduction

After the discovery of chaos there arose lots of questions regarding the routes to the development of chaotic behaviour. What is the scenario behind the transition from regular behaviour to the chaotic behaviour as the control parameter is varied ? Are there any universal patterns or sequences for this transition ? Different routes to chaos have been reported in the chaos literature. The period doubling route to chaos, quasi periodic route to chaos, intermittency route to chaos, crisis induced intermittency route to chaos *etc.* are some of the widely accepted scenarios. These routes to chaos are the ways in which the laminar flow loses stability and becomes chaotic. Manneville and Pomeau (1979) introduced the intermittency route to chaos in the Lorenz equations. The so-called intermittency occurs when nearly regular behaviour (laminar flow) is intermittently interrupted by chaotic outbreaks (bursts) at irregular intervals. As the control parameter is increased (or decreased), the strength of the chaotic burst also increases and finally the system ends up in fully chaotic behaviour. One of the routes to chaos is the class I intermittency route to chaos.

4.2 Class I intermittency

The scenarios describing the creation or destruction of a chaotic attractor as a parameter is varied are also important in the field of dynamical systems. It is interesting to see what happens in a dynamical system as the control parameter is changed. If the trajectories in phase space of a dynamical system before and after a specified value of the control parameter are qualitatively different, then the qualitative difference is called a bifurcation. The transition from a stable periodic orbit to an unstable periodic orbit is a qualitative change. If branches of stable and unstable periodic orbits coincide at a particular value of the control parameter, then such a bifurcation is called cyclic-fold bifurcation. Assume that a dynamical system is under the influence of cyclic-fold bifurcation at a particular value (say λ) of the control parameter in such a way that the attractor is a limit cycle for all values of the control parameter less than the particular value, λ. At the same time it is impossible to determine the system behaviour for values greater than λ by local bifurcation analysis.

The behaviour after bifurcation can be analyzed by numerical computations. One possibility is that the attractor before bifurcation may be switched onto a new attractor in which the old attractor will be a proper subset of the new attractor. The analysis of the time series obtained by computer simulation shows that an orbit in the attractor after the bifurcation point stays back near the destroyed limit cycle for a long time and is interrupted by chaotic bursts. The orbit near the destroyed limit cycle is called the laminar phase and the chaotic burst between the laminar phases is called the turbulent phase. As the control parameter is increased (or decreased in some other cases) the average time spent by the chaotic bursts in the attractor tends to infinity the attractor become fully chaotic. This type of transition from periodic behaviour to chaotic behaviour via a cyclic-fold bifurcation is called the intermittency route to chaos of class I. The existence of the mechanism that reinjects the trajectory lying in the chaotic bursts to the vicinity of the limit cycle is necessary for the intermittency route to chaos. If not, the orbit will never revisit the vicinity of the limit cycle. A detailed treatment of different types of intermittency routes to chaos is available in the literature (Pomeau and Manneville 1980; Hilborn 1994; Nayfeh and Balachander 1995). We observed a new type of class I intermittency in the dynamics of periodically forced spheroids in simple shear flow which is discussed in the following section.

4.3 Results on the new class I intermittency

In this section we report a physically realizable system in which the possibility of an interesting and novel type of Class I intermittency, namely, a non hysteretic form of Class I intermittency with nearly regular behaviour interrupted by chaotic outbreaks (bursts) with nearly regular reinjection period is demonstrated. The bursting process was irregular in almost all previous computational and experimental studies of Class I intermittency. Price and Mullin (1991) have observed experimentally a similar type of phenomenon in which a hysteretic form of intermittency with extreme regularity of the bursting is observed. The system described in this section appears to be one of the very few ODE systems describing a physically realizable system showing a non hysteretic form of Class I intermittency with nearly regular behaviour interrupted by chaotic outbreaks (bursts) with nearly regular reinjection period. We also present appropriate return maps to explain the behaviour observed in this work. The system also shows certain interesting features such as new scaling behaviour away from the onset of intermittency and the number of the bursts during a particular realization varying smoothly with the control parameter. We discuss and compare the model of Class I intermittency with the new type of intermittency. The comparison with the theoretical predictions of Class I intermittency shows scaling typical of Class I intermittency. The average length of the burst also scales with the control parameter with zero slope.

The phenomenon of Class I intermittency with nearly constant reinjection period, we observed in this work was obtained in the parametric regime $10.6 \leq k_2 \leq 12.44$. We report the results for $0.01 \leq \theta \leq 40°$ and for all values of ϕ in steps of $20°$ which results in 15 initial conditions for a given parameter value. The evolution of the particle upto 2150 cycles was computed and the results are presented for $r_e = 1.6, \omega = J = 2\pi(r_e + r_e^{-1})$, and $k_1 = k_3 = 0,$. For the trajectory we evaluated 100 points in each cycle which resulted in 215000 points and deleted 200000 points (2000 cycles) as transients. It was found to be sufficient to concentrate on the remaining 15000 points of the trajectory to study the new behaviour, because when the number of iterations was doubled the only change in the observed behaviour was a doubling in the number of bursts with no other change in the dynamics. We expect the greatest complexity of the solutions of the above equations for this particular choice of parameters, since k_2 is responsible for the greatest opposition to the hydrodynamic torque due to the imposed shear flow field. For $k_2 = 0$, Jeffery's results are reproduced and all solutions of the equations starting from different initial conditions tend towards a fixed point in the stroboscopic plot. Upon changing k_2 we observed a number of chaotic regimes of the parameter k_2 as well as a number of regular regimes in between the chaotic regimes. The phenomena we wish to report in this work lies in the parameter regime $10.6 \leq k_2 \leq 12.44$. We also confine ourselves to the value of $0.01 \leq \theta \leq 40°$ and $0° \leq \phi \leq 90°$ in steps of $20°$. This choice of initial conditions and range of k_2 results in interesting behaviour with clear evidence for the new behaviour, namely Class I intermittency with nearly constant reinjection period. For all values of k_2, θ and ϕ within the above range, the system shows similar behaviour.

The analysis of the time series and the attractor shows the existence of a tangent bifurcation which leads to the novel behaviour of Class I intermittency with nearly constant reinjection period. The maximum Lyapunov exponents of both the time series of the bursts (denoted by $\mathbf{X}$) and the laminar phase (denoted by **Y**) were positive and entirely different. Fig. 4.1 shows a superposition of the trajectories corresponding to the laminar phase and the bursts and Fig. 4.2 shows the corresponding time series. During the bursts, the trajectory moves away from the vicinity of the laminar region as is evident from Fig. 4.1. The maximum Lyapunov exponent for the bursts was nearly constant for all θ, ϕ and k_2 considered and was equal to 0.21, indicating that the bursts show the same type of chaotic behaviour everywhere in the system. At the same time the maximum Lyapunov exponent for the laminar phase decreases with increasing k_2 and finally ends up in a periodic behaviour at $k_2 = 12.46$. The system is more sensitive to initial conditions near the chaotic bursts (Type **X**) and less sensitive to initial conditions near the laminar phase (Type **Y**) as can be seen from the time series of trajectories for slightly different initial conditions given in Fig. 4.3.

The existence of the intermittency behaviour in the system was further confirmed by solving the evolution equations 2.18 and 2.21 in single and double precision. The trajectories obtained from equation 2.21 in single and double precision and from equation 2.18 in single precision for the same set of parameters and initial conditions are given in Fig. 4.4. There will be small changes in the evolutions obtained from the numerical computations in single and double precision due to round off error. Hence if the orbit is in a chaotic attractor, the time series obtained here will diverge exponentially as is evident from Fig. 4.4. However all the time series obtained show the same intermittency behaviour. As k_2 is increased near the onset of tangent bifurcation the bursts remain chaotic with nearly the same Lyapunov exponent equal to 0.21. The Lyapunov exponents of the laminar phase were small compared to that of the bursts and decreased slowly and became nearly zero as k_2 is increased near the onset of tangent bifurcation as can be seen from Table. 4.1. Table 4.1 shows that the maximum Global Lyapunov exponent (GE) on the average lies between the maximum Local Lyapunov exponent (LE) of the laminar phase and the chaotic

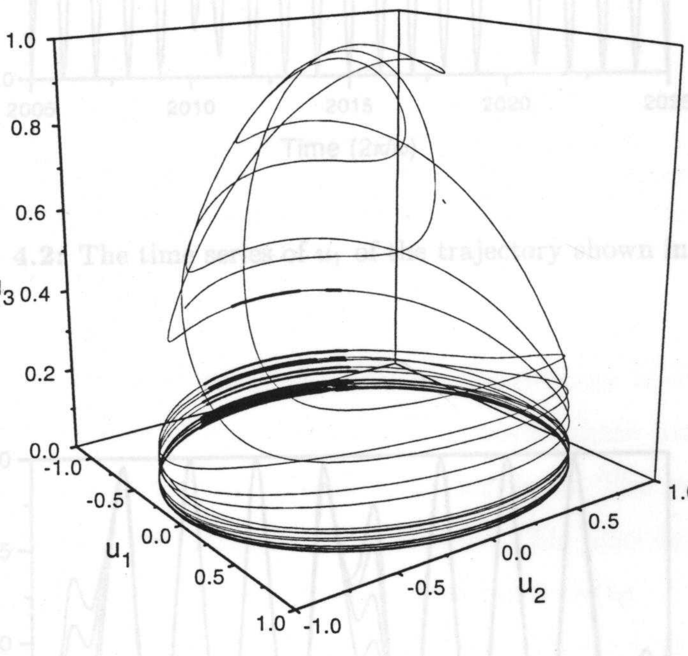

Figure 4.1: Typical phase space trajectory of the attractor showing the laminar phase as well as the chaotic burst for $k_2 = 12.2$, initial conditions $\theta = \phi = 20°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$.

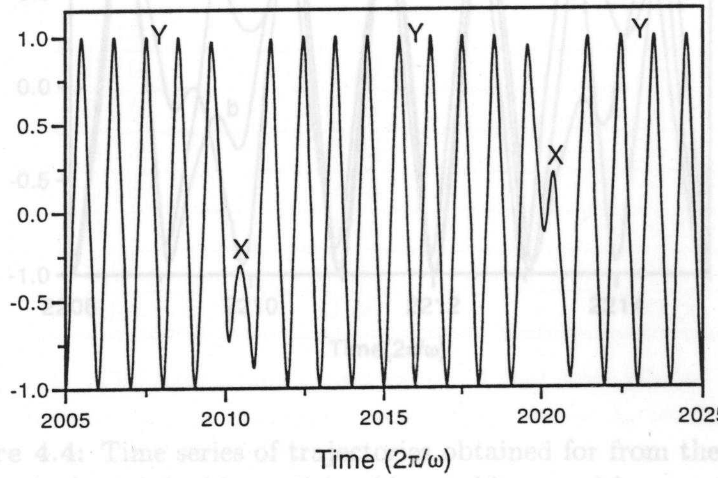

Figure 4.2: The time series of u_1 of the trajectory shown in Fig. 4.1

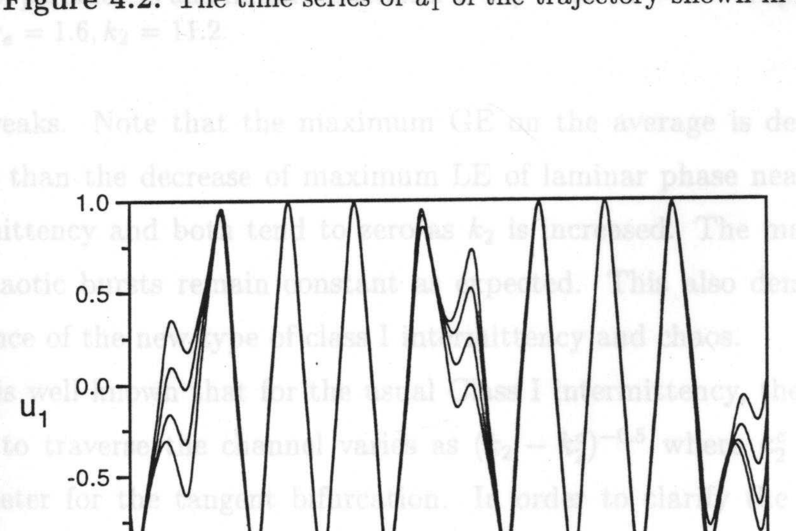

Figure 4.3: Time series of different trajectories for $k_2 = 11.2$, slightly different initial conditions $\theta = 20.0°$, $20.1°$, $20.2°$, $20.3°$, $\phi = 20°$, $\omega = J = 2\pi(r_e + {r_e}^{-1})$, $r_e = 1.6$.

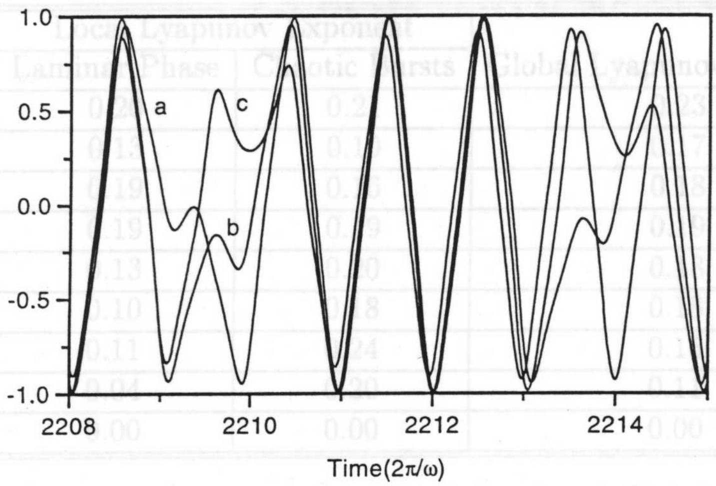

Figure 4.4: Time series of trajectories obtained for from the equations 2.21 in single and double precision (denoted by a and b respectively) and from the equations 2.18 in single precision (denoted by c) for the same set of parameters and initial conditions. $\theta = \phi = 20°$, $\omega = J = 2\pi(r_e + {r_e}^{-1})$, $r_e = 1.6, k_2 = 11.2$.

out breaks. Note that the maximum GE on the average is decreasing more slowly than the decrease of maximum LE of laminar phase near the onset of intermittency and both tend to zero as k_2 is increased. The maximum LE of the chaotic bursts remain constant as expected. This also demonstrates the existence of the new type of class I intermittency and chaos.

It is well known that for the usual Class I intermittency, the average time taken to traverse the channel varies as $(k_2 - k_2^c)^{-0.5}$ where k_2^c is the critical parameter for the tangent bifurcation. In order to clarify the nature of the intermittency transition to chaos, we computed the scaling of the average length of the laminar phase with the excess of the critical value of the parameter over the control parameter very near to the tangent bifurcation at $k_2 = 12.44$ (in steps of 0.02) and found the exponent to be -0.60 as shown in Fig. 4.5. The scaling behaviour near the onset of bifurcation shows that the observed transition is a typical Class I intermittency. This was further confirmed by the

k_2	Local Lyapunov Exponent		Global Lyapunov Exponent
	Laminar Phase	Chaotic Bursts	
10.6	0.20	0.21	0.23
11.0	0.13	0.19	0.17
11.4	0.19	0.16	0.18
11.8	0.19	0.19	0.19
12.0	0.13	0.20	0.18
12.2	0.10	0.18	0.18
12.4	0.11	0.24	0.15
12.44	0.04	0.20	0.11
12.46	0.00	0.00	0.00

Table 4.1: The maximum values of the Local and Global Lyapunov exponent obtained for different values of k_2, $k_1 = k_3 = 0, \omega = J = 2\pi(r_e + r_e^{-1}), r_e = 1.6, \theta = \phi = 20°$

superposition of the return map of ϕ in the dynamics and the appropriate map, $\phi_{n+1} = \phi_n + {\phi_n}^2 + \epsilon$. Fig. 4.6 shows that the two maps coincide near the tangent bifurcation, where the map R_2 obtained from $\phi_{n+1} = \phi_n + {\phi_n}^2 + \epsilon$ is presented for ϵ equal to zero.

For detailed study we confined our results to $\theta = \phi = 20°$ and $10.6 \leq k_2 \leq 12.44$. We analyzed the return maps of ϕ by varying the control parameter for comparatively large steps of k_2, namely 0.2. In the return maps the points corresponding to the onset of the laminar phase fell on the curve between the points marked as A and B and remainder of the return maps correspond to the intermittent bursts as given in the Fig. 4.7. Fig. 4.7 shows the return maps of ϕ of the Poincaré Sections (stroboscopic plot) of 5000 iterations of the system for one set of initial conditions and value of k_2. Some points of a typical return map are plotted by joining all the points to establish the fact that the reinjection period is nearly constant and the plot is given in Fig. 4.8. The portion of the return map shown in Fig. 4.8 corresponding to the region between A and B corresponds to the laminar phase. An examination of Fig. 4.8 shows that once the system leaves the laminar phase, it is reinjected into the neighbourhood of

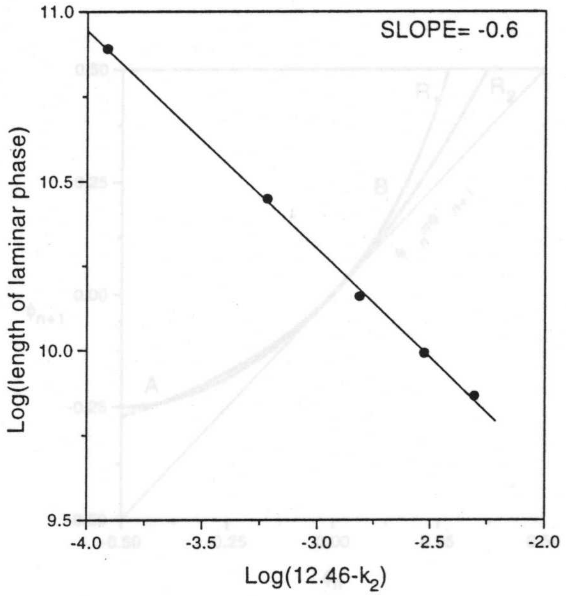

Figure 4.5: Plot of logarithm of average length of laminar phase as a function of logarithm of excess of control parameter showing scaling typical of Class I Intermittency.

the tangent bifurcation in approximately two or three periods. The reinjection time remains nearly unaffected by changes in control parameter whereas, as expected, the average length of the laminar phase increases as the system moves away from the tangent bifurcation.

The portion of the return maps shown in Fig. 4.7, which correspond to regions away from the laminar phase, namely the smooth curve below the line $\phi_{n+1} = \phi_n$ and the two intersecting straight lines, namely C_1, C_2 and C_3 and is explored during the bursts lies less often on these curves once the average length of the laminar phase increases. As a result, fewer points of the stroboscopic return map lie on these curves as k_2 increases. The points of intersection D and E with the line $\phi_{n+1} = \phi_n$ were confirmed to be non-attracting by starting the computation with initial conditions near the points of intersection of the curve D and E with the line $\phi_{n+1} = \phi_n$. To further establish the observation

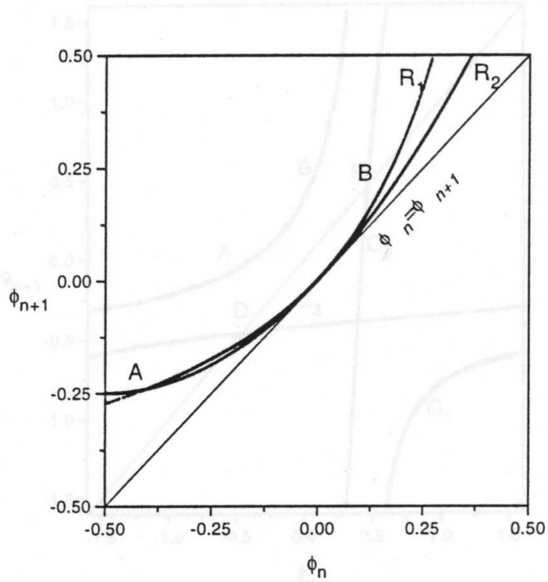

Figure 4.6: Superposition of the observed return map (R_1) obtained from Eq.2.21 at the point of tangent bifurcation and the classic map (R_2) exhibiting Class I Intermittency, where the return map, R_1 is presented for $k_2 = 12.44$, initial conditions $\theta = \phi = 20°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$

of nearly regular reinjection process and the process is unaffected by variation in the control parameter, the return maps of ϕ for different values of k_2 were analyzed along with the line $\phi_{n+1} = \phi_n$. It can be seen in Fig. 4.9 that the reinjection process is unchanged by the variation in the control parameter.

Another interesting feature of this system is that the average length of the laminar phase varies smoothly with the excess of the critical parameter over the control parameter even when the system is relatively far away from the onset of intermittency. This was confirmed numerically by checking that the logarithm of the average life time of the laminar phase scaled linearly with the logarithm of $12.6 - k_2$ with a slope of -1.15 as shown in the Fig. 4.10. Interesting scaling behaviour is observed near the critical value of the control parameter as shown in Fig. 4.11 where the length of laminar phase scales linearly with $12.4 - k_2$.

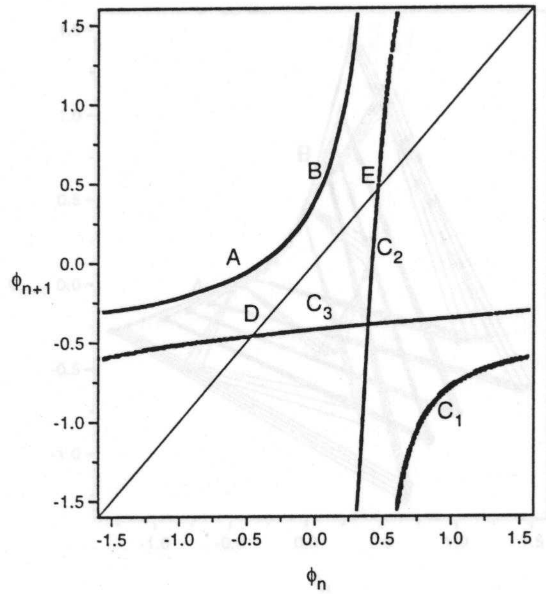

Figure 4.7: The return map ϕ for 5000 points of Poincaré section (stroboscopic plot) of the system for $k_2 = 11.0$, initial conditions $\theta = \phi = 20°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$.

The number of the bursts as well as the number of occurrences of the laminar phase decreases as k_2 is increased. Further, the average length of the bursts is nearly constant and the average length of the laminar phase increases as k_2 increases. It is also found that length of the burst scales with control parameter near $k_2 = 12.4$ with zero slope. The number of bursts as well as the number of occurrences of the laminar phase decreases smoothly as k_2 increases. As a result, scaling behaviours of the average length of the laminar phase with $k_2 - 10.4$ are obtained. Class I intermittency behaviour with nearly constant reinjection period continued upto a value of k_2 equal to 12.44. Beyond k_2 equal to 12.44 this phenomenon disappeared and the system response becomes periodic.

The feature of the system studied which we feel is interesting to the nonlinear dynamics community is the existence of Class I intermittency with nearly constant reinjection period. This implies that the length of the burst between

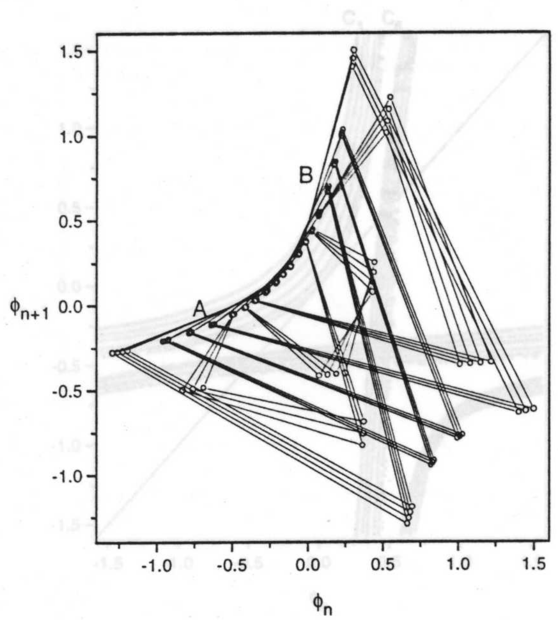

Figure 4.8: The return map ϕ for 100 connected points of Poincaré section (stroboscopic plot) of the system for $k_2 = 12.2$, initial conditions $\theta = \phi = 20°, \omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$ showing that the reinjection period is nearly constant.

two laminar phases is nearly constant. An analysis of the return maps for ϕ shown in Figs. 4.7 and 4.8 indicates that once the system leaves the neighbourhood of the tangency, it comes back to the neighbourhood of the tangency in nearly the same number of iterations. At k_2=12.44 the attractor is nearly regular. The system was purely periodic with period 6 at k_2=12.46.

Numerical evidence for Class I intermittency with nearly constant reinjection period has been presented near the onset of tangent bifurcation. New scaling behaviour which governs how the average life time of the laminar phase scales with control parameter away from the onset of intermittency is presented. The return maps of the stroboscopic plot for different values of k_2 given in Fig. 4.9 provide an explanation for this behaviour. The present system is one of the very few physically realizable systems which shows this phenomenon of

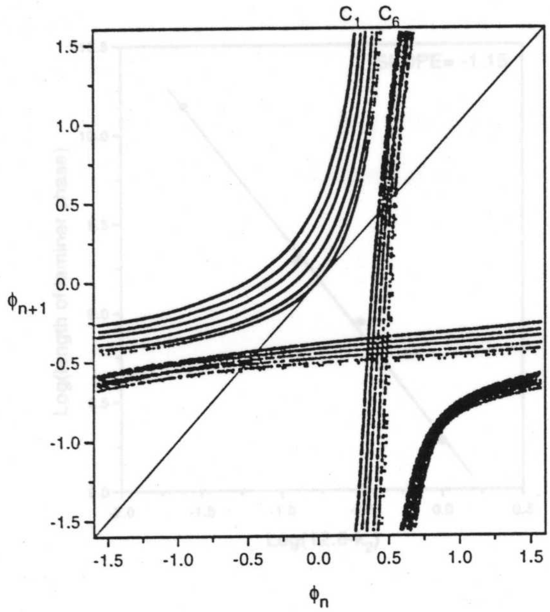

Figure 4.9: Superposition of return maps of ϕ for different values for k_2 =10.6, 11.0, 11.4, 11.8, 12.2, 12.44, initial conditions $\theta = \phi = 20°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$. Curve C_1 is for k_2 =10.6, Curve C_6 is for k_2 =12.44 with the other curves representing values of k_2 in ascending order.

Class I intermittency with nearly constant reinjection period. Since this problem is technologically important as is evident from the literature cited in this work, the existence of the new behaviour in this system and the existence of the interesting transient behaviour may have important practical consequences, when operating near the regime considered in this work where this phenomenon may not be recognized as leading to chaos. An analysis of the original model equations to explain to some extent the behaviour presented in this work could be carried out. Price and Mullin (1991) and Aubry *et al.* (1988) have reported similar behaviour with other model equations. They have performed some preliminary analysis of their model equations in an attempt to explain the behaviour observed. A similar analysis of our model equations would be considerably more involved since our equations are non-autonomous and would

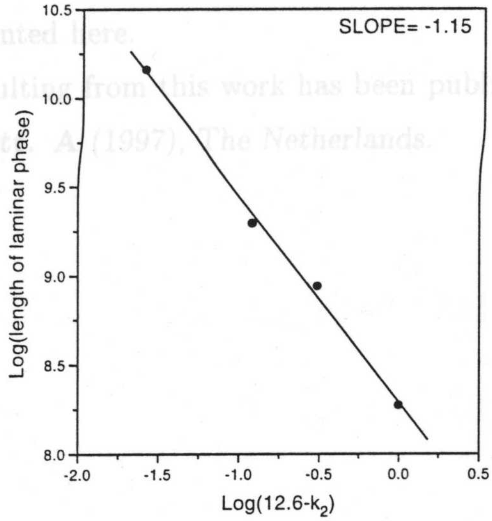

Figure 4.10: Plot of logarithm of average length of laminar phase as a function of logarithm of excess of control parameter showing scaling away from the tangent bifurcation.

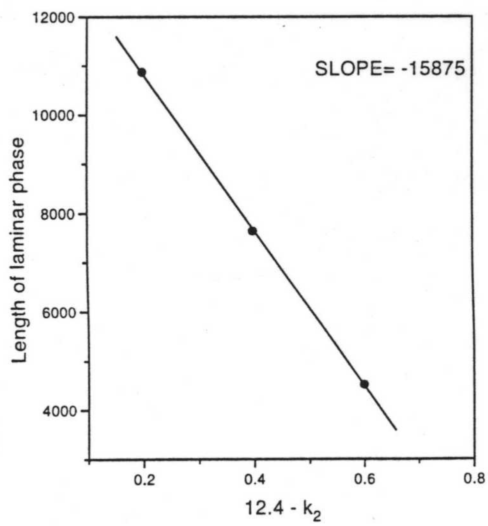

Figure 4.11: Plot of average length of laminar phase as a function of excess of control parameter showing linear scaling away from the tangent bifurcation.

not necessarily throw more light on the results than the analysis based on the return maps presented here.

One paper resulting from this work has been published in an international journal **Phys. Lett. A** (1997), The Netherlands.