Chapter 5

The Theory of Dynamics with Control

5.1 Introduction

The fact that a chaotic solution eliminates the possibility of long term prediction of system behaviour induced many reports in the literature of either quenching chaos or controlling chaos (Ott *et al.* 1990 and Ditto *et al.* 1995). Since chaotic attractors have embedded within them a dense set of unstable periodic orbits, any one of the unstable periodic orbits can be stabilized to obtain otherwise unattainable system behaviour. The essential idea is that a chaotic system explores a relatively large region of state space and the system can be brought to a desired stable state to improve the performance of the separation technique by a suitable control algorithm. The first method (OGY) of control of chaos proposed by Ott *et al.* (1990) generated appreciable interest in the literature of chaos. Thereafter, a large number of algorithms for controlling chaos have been reported in the literature (Ott and Spano 1995, Rhode *at al.* 1995, Christini and Collins 1995 and references therein). Broadly speaking there are two classes of algorithms for controlling chaos, namely (a) *Feedback Methods* (Ott *et al.* 1990) and (b) *Non-feedback Methods* (Güémez *et al.* 1994). The first method needs appreciable information about system behaviour but is comparatively simple to implement experimentally. The second method is theoretically simple and

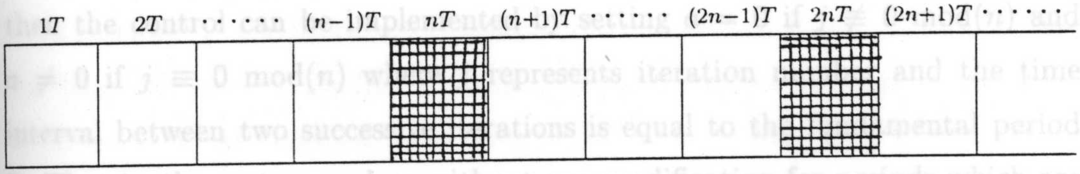

Figure 5.1: The schematic representation of the novel control strategy presented in this work. The control is active during the period corresponding to the shaded area and not active during the period corresponding to the blank area, where each box stands for the period. T of length $2\pi/\omega$.

hence it is easy to implement on a computer whereas it is difficult to implement experimentally in many systems.

The algorithm suggested in this work needs very little information about system behaviour but is rather easy to implement experimentally. This algorithm can be classified as a *Non-feedback control method.* Rajasekar and Lakshmanan (1993) have investigated the applicability and effectiveness of various approaches of controlling chaos in the BVP oscillator. Suppression of chaos by periodic parametric perturbations in an experimental set up of a Duffing oscillator is also reported in the literature by Fronzoni *et al.* (1991). In the method proposed in this section, the control parameter is perturbed for one period at fixed intervals of every integral multiple of the fundamental period. In the methods considered earlier in the literature (Lakshmanan and Murali 1996), the parameter was perturbed continuously rather than at fixed intervals. To implement the control strategy reported in the section 5.2, we apply a constant external force in addition to the periodic force for the duration of one period after every $n-1$ periods as shown in Fig. 5.1, where the length of one period (T) is equal to length of the fundamental period of the periodic force. If the system is perturbed after $n-1$ periods by applying an additional constant force, the system is found to be stabilized in a periodic orbit with a period equal to nT or with a period equal to an integral multiple of nT.

For example, if a represents the magnitude of the additional constant force, then the control can be implemented by setting $a = 0$ if $j \not\equiv 0 \text{ mod}(n)$ and $a \neq 0$ if $j \equiv 0 \text{ mod}(n)$ where j represents iteration number and the time interval between two successive iterations is equal to the fundamental period T. That is, the system evolves without any modification for periods which are not multiples of n and with modification for periods which are multiples of n. A schematic representation of the control strategy is given in Fig. 5.1. The choice of n depends on the period-m solution we want to stabilize. The integer n can be either m or a divisor of m depending on the value of a and the choice of n.

5.2 A new algorithm for control of chaos

To incorporate the above idea of controlling the dynamics of the system, we modify Eq. 2.21 governing the dynamics of the system by introducing a constant force in addition to the periodic force along the direction of the periodic force for one period at the end of n forcing periods. Thus the resultant torque induced on the particle is given by $\boldsymbol{L} = (\boldsymbol{k}\cos(\omega t) + \boldsymbol{k}') \times \boldsymbol{u}$. Let k_1', k_2' and k_3' be the x, y, z components of $\boldsymbol{k}'$. After scaling all quantities appearing in the equations similarly as explained in the section 2.7, the Eq. 2.21 can be written as

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{P}{2}\sin 2\theta \sin 2\phi + R\left[(\cos\theta\cos\phi\, k_1 + \cos\theta\sin\phi\, k_2 - \sin\theta\, k_3)\cos(\omega t)\right] \\
&\quad + R\left[(\cos\theta\cos\phi\, k_1' + \cos\theta\sin\phi\, k_2' - \sin\theta\, k_3')\right] \\
\frac{d\phi}{dt} &= P\cos 2\phi - Q \\
&\quad + \frac{R}{\sin\theta}\left[(-\sin\phi\, k_1 + \cos\phi\, k_2)\cos(\omega t) + (-\sin\phi\, k_1' + \cos\phi\, k_2')\right] \qquad (5.1)
\end{aligned}$$

It is noted that the modified system of equations 5.1 reduces to Eq. 2.21 when $k_1' = k_2' = k_3' = 0$.

The application of an additional external forcing (constant) to control chaotic systems evolving under the effect of external forcing (periodic) has been reported in the literature (Lakshmanan and Murali 1996). In this method the constant force is applied continuously. Our control strategy involves the application of the constant force for a period of finite length, T after a period of length, $(n-1)$T (where T=$2\pi/\omega$ is the fundamental period of the periodic forcing). Then, we lift the control for a period of finite length, $(n-1)$T and thereafter we apply the constant force of the same magnitude for a period of length, T and again we lift the control for a period of finite length, $(n-1)$T. This process is repeated. Under this process, the system is allowed to evolve according to the system of Eq. 2.21 upto the $(n-1)^{\text{th}}$ forcing period and evolve according to the system of Eqs. 5.1 at the n^{th} forcing period. This process is repeated every n^{th} period. While solving the Eqs. 5.1, this idea can be implemented by setting $k'_1 = k'_2 = k'_3 = 0$ if $j \not\equiv 0 \bmod(n)$ and $k'_1 = k'_2 = k'_3 \neq 0$ if $j \equiv 0 \bmod(n)$ where j represents iteration number and $T = 2\pi/\omega$ is the time interval between two successive iterations. We note that the above equations for $\dot{\theta}$ and $\dot{\phi}$ decouple in the absence of k_1, k'_1 and k_2, k'_2. Hence the presence of an external force field with either k_1 or k_2 is necessary to obtain chaotic solutions in this system. In our calculations we kept $k_1 = k_3 = k'_1 = k'_3 = 0$. In almost all cases the system is stabilized to a periodic orbit with period n if the control is applied throughout the n^{th} period. In some cases the system is stabilized to a periodic orbit of period equal to an integral multiple of n. On the other hand if the control is applied continuously, we loose the flexibility of controlling the system to an orbit of desired period.

5.3 Results on separation technique with control

We have analyzed the properties of the system without control in chapter 3. We tentatively identified chaotic regimes of the parameter k_2 keeping $k_1 = k_3 = 0$. As a first step in analyzing the properties of the equations derived in this work, we set $k_1 = k_3 = 0$. in equations 2.21 and varied k_2 for particles of different aspect ratio within the range of r_e ranging from 0.2 to 2.0 in steps of 0.2 and kept ω equal to $J = 2\pi(r_e + r_e^{-1})$. We ran the program for 2500 points of the Poincaré section (stroboscopic plot) and deleted the first 2250 points to remove the transients. We started the trajectory with the initial conditions $\theta = \phi = 45°$. For each trajectory we evaluated 100 points in each cycle which resulted in 25000 points of the trajectory after the transients are removed. We selected $k_2 = 12.0$ and computed the solution of the equations for different r_e ranging within 0.2 to 2.0 in steps of 0.2. For $k_2 = 12.0$ the system given by Eq. 2.21 behaved chaotically for all the r_e considered except for r_e equal to 1.8 and 2.0. The Lyapunov exponents of the attractors were evaluated and found to be positive. The attractor and time series of a typical trajectory are shown in Fig. 5.2 and Fig. 5.3 for the case of r_e=1.2.

In chapter 3, it was reported that the results of the computations are very sensitive to the aspect ratio of the particle in some parameter regimes. In the case of constant external fields in the same parametric regimes we obtained regular behaviour. For $r_e > 1.0$, we obtained nearly the same fixed point for all initial conditions in the case of a constant force field. This indicates that in the sample application considered in this work, a periodic force field is necessary to effect particle separation for particles with $r_e > 1.0$ as explained in section 3.3. Tables. 3.1, 5.1 and 5.2 give a sample of the results obtained for zero force, a constant force of amplitude $k_2 = 12.0$ and a periodic force of amplitude $k_2 = 12.0$ respectively. In this case independent separation of particles is possible only for particles of aspect ratio r_e equal to 1.2. The existence of chaotic dynamics in this system allows control of its dynamics to a desired orbit and thus suggests the possibility of better separation of particles for almost all particles of aspect ratio ranging within 0.2 to 2.0. This is difficult to obtain in the case of regular behaviour or chaotic behaviour without control.

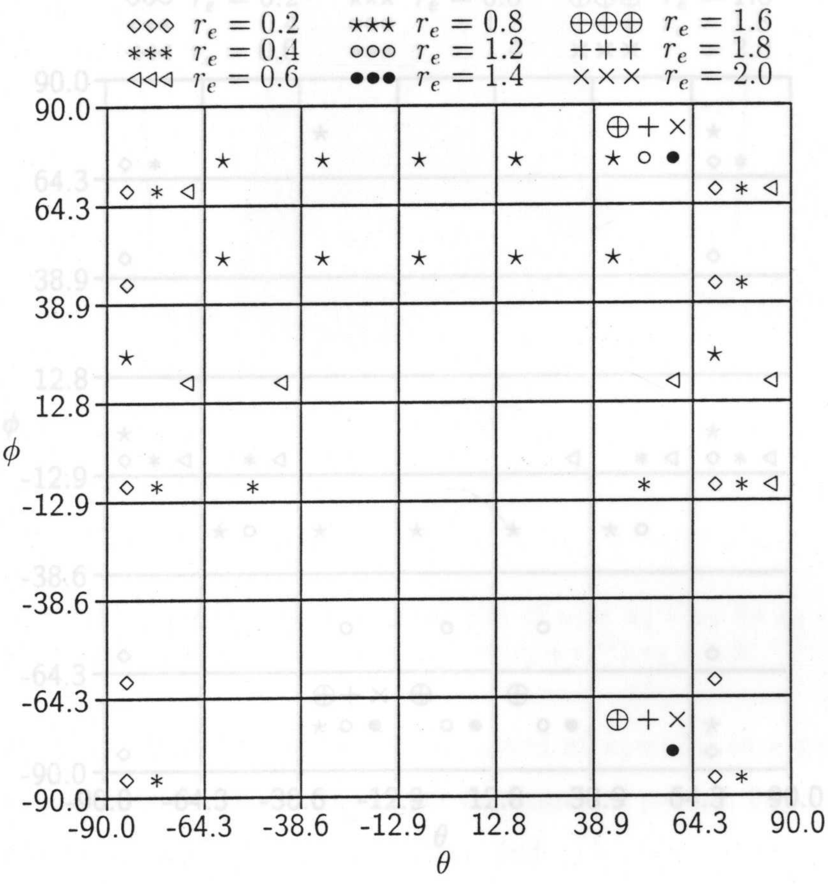

Table 5.1: Distribution of evolution of initially uniformly distributed particles of different aspect ratios for the case $k_2 = 12.0$, $\omega = 0$, $5 \leq l_1 \leq 49$ and $100 \leq l_2 \leq 1000$, where l_1 is the total number of particles in the grid on the average and l_2 is the total number of occurrences of the grid.

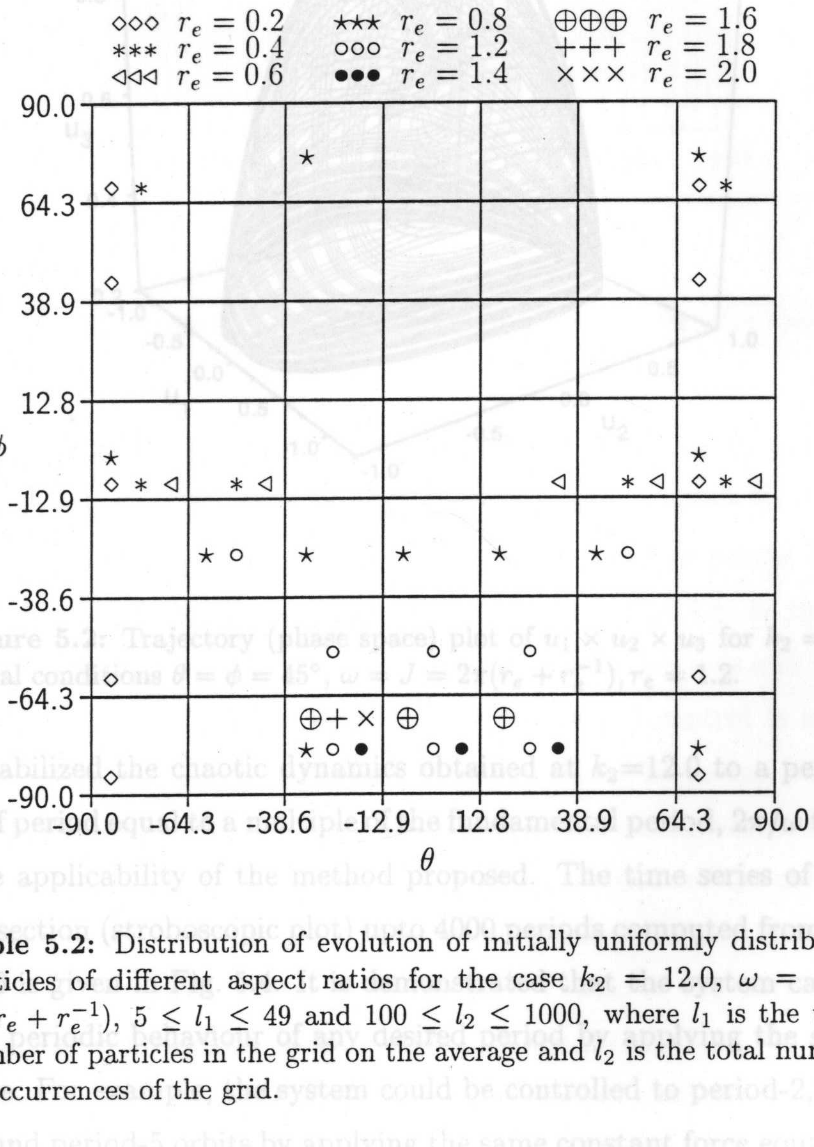

Table 5.2: Distribution of evolution of initially uniformly distributed particles of different aspect ratios for the case $k_2 = 12.0$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $5 \leq l_1 \leq 49$ and $100 \leq l_2 \leq 1000$, where l_1 is the total number of particles in the grid on the average and l_2 is the total number of occurrences of the grid.

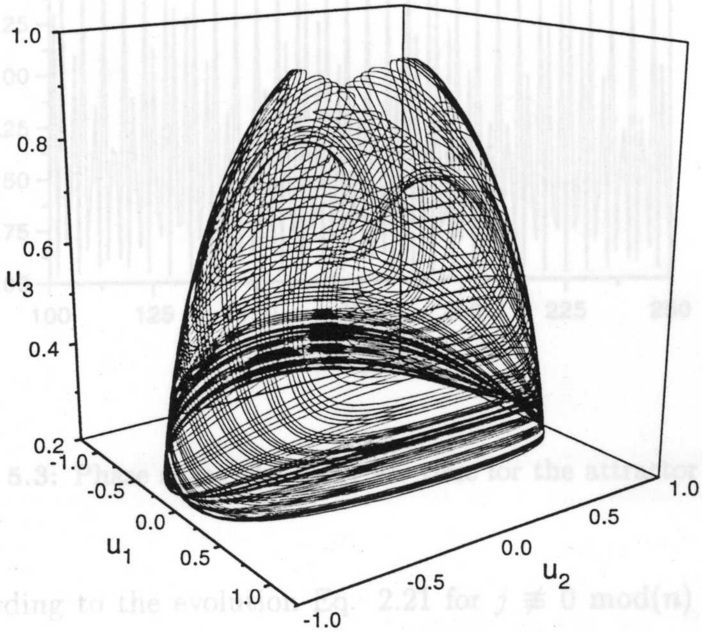

Figure 5.2: Trajectory (phase space) plot of $u_1 \times u_2 \times u_3$ for $k_2 = 12$, initial conditions $\theta = \phi = 45°$, $\omega = J = 2\pi(r_e + r_e^{-1}), r_e = 1.2$.

We stabilized the chaotic dynamics obtained at k_2=12.0 to a periodic behaviour of period equal to a multiple of the fundamental period, $2\pi/\omega$ to demonstrate the applicability of the method proposed. The time series of u_2 of the Poincaré section (stroboscopic plot) upto 4000 periods computed from Eq. 2.21 for r_e=1.6 is given in Fig. 5.4. It is demonstrated that the system can be controlled to periodic behaviour of any desired period by applying the same constant force. For example, the system could be controlled to period-2, period-3, period-4 and period-5 orbits by applying the same constant force equal to k_2'=5 as can be seen from Figs. 5.5. This is obtained by setting $k_1' = k_2' = k_3' = 0$ if $j \not\equiv 0 \bmod(n)$ and $k_1' = k_3' = 0$, $k_2' = 5.0$ if $j \equiv 0 \bmod(n)$ in the evolution of Eqs. 5.1 when the system is to be stabilized in a period-n orbit. The system

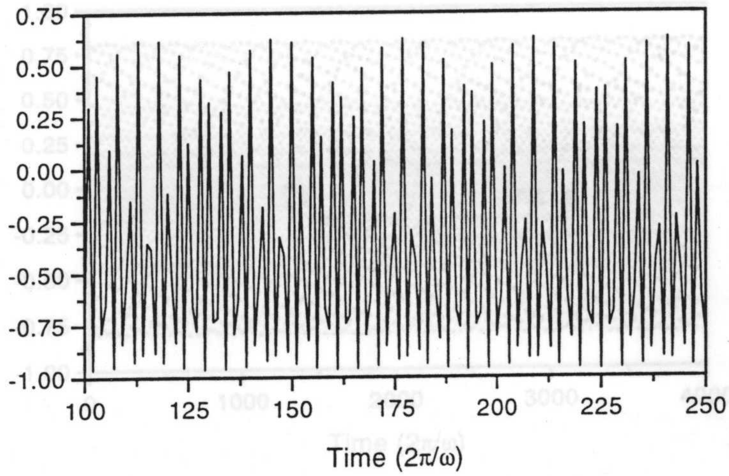

Figure 5.3: Phase space plot of u_1 vs. time for the attractor shown in Fig. 5.2.

evolves according to the evolution Eq. 2.21 for $j \not\equiv 0 \bmod(n)$ and according to the evolution Eqs. 5.1 for $j \equiv 0 \bmod(n)$. It is also possible to stabilize the system to qualitatively different periodic orbits of the same period by suitably changing the control parameter k_2' as can be seen from Fig. 5.6. In this example the control technique is applied after 1500 fundamental periods and the system is followed upto 3000 periods with control. Once the control is applied the system stabilizes rapidly to the appropriate periodic orbit as can be seen from the sample figures in Fig. 5.5. The magnitude of the perturbation required to stabilize the system was small compared to the magnitude of the periodic force for all aspect ratios except r_e= 0.2, 0.4 and 0.6. The magnitude of the constant force required for control depends on the aspect ratio of the particle and the desired period, n for aspect ratios less than or equal to 0.6.

One important advantage of the control algorithm outlined in this work is the possibility of switching over to different periodic solutions during a given run. This implies that a system in chaotic dynamics can be stabilized to one particular periodic orbit for a given time and to another periodic orbit of en-

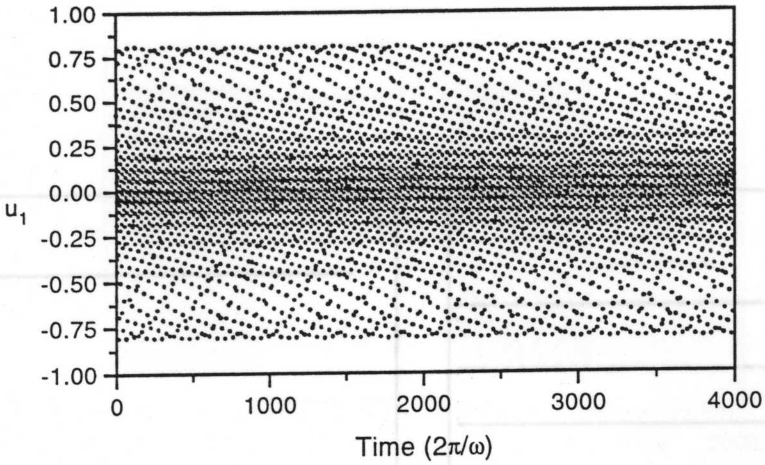

Figure 5.4: Trajectory plot of u_2 vs. time at every intersection of the trajectory with the Poincaré plane for $k_2 = 12$, initial conditions $\theta = \phi = 45°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$.

tirely different period after a given time as shown in Fig. 5.7. In this example the control parameter $k_2' = 5.0$ is applied between 1001 and 2000 periods to stabilize the system to a period-2 orbit. Then control is lifted between 2001 and 3000 periods and again applied between 3001 and 4000 periods to stabilize the system to a period-3 orbit. Control was removed again between 4001 and 5000 periods. Thus the system oscillates in a period-2 orbit between 1001- 2000 periods and then oscillates in a period-3 orbit between 3001- 4000 periods as can be seen from Fig. 5.7. This figure also reveals the fact that once the control is lifted, the system returns to a chaotic state. Another advantage which is important from the point of view of the sample application proposed in this work is the possibility of changing the periodic behaviour to another orbit of the same period by changing the magnitude of the applied constant force. This allows us to bring a particle having a definite aspect ratio to a desired orbit by changing control parameters. It is also noted that all initially uniformly distributed particles of a given aspect ratio within the range $0.8 \leq r_e \leq 2.0$ can be concentrated in a given grid by applying a periodic force with control as can

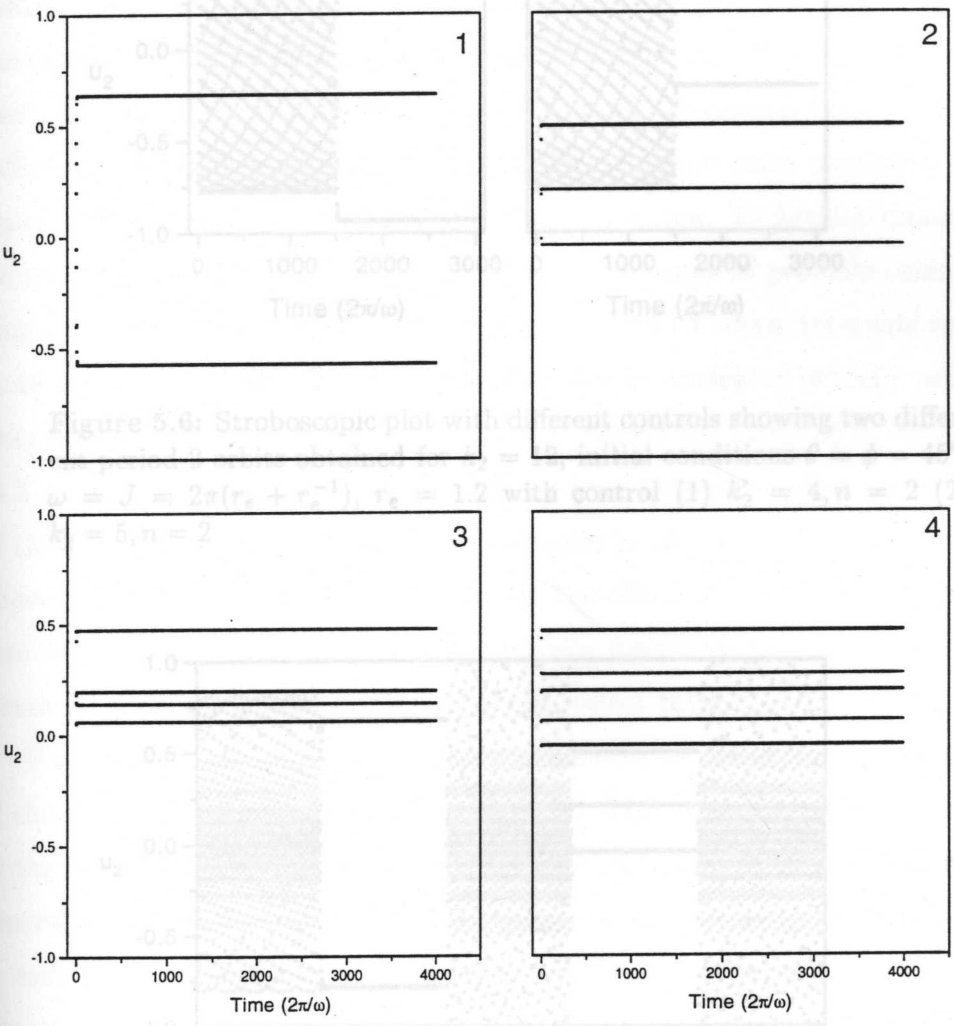

Figure 5.5: Stroboscopic plot with control for $k_2 = 12$, $k_2' = 5.0$, initial conditions $\theta = \phi = 45°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6$ showing (1). a period-2 solution with $n = 2$ (2). a period-3 solution with $n = 3$ (3). a period-4 solution with $n = 4$ (4). a period-5 solution with $n = 5$

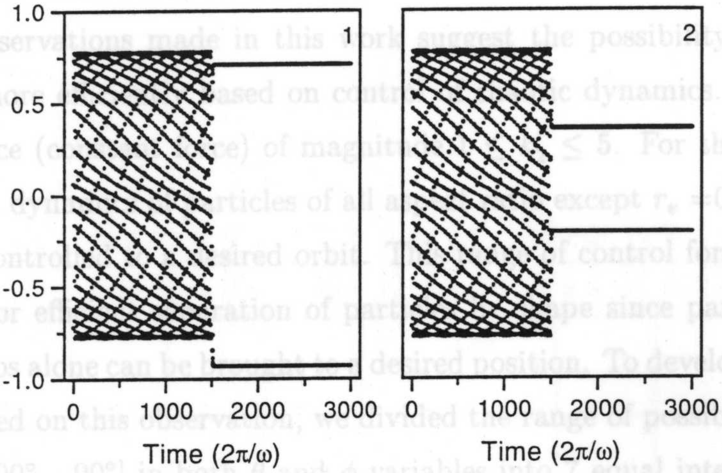

Figure 5.6: Stroboscopic plot with different controls showing two different period-2 orbits obtained for $k_2 = 12$, initial conditions $\theta = \phi = 45°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.2$ with control (1) $k_2' = 4, n = 2$ (2) $k_2' = 5, n = 2$

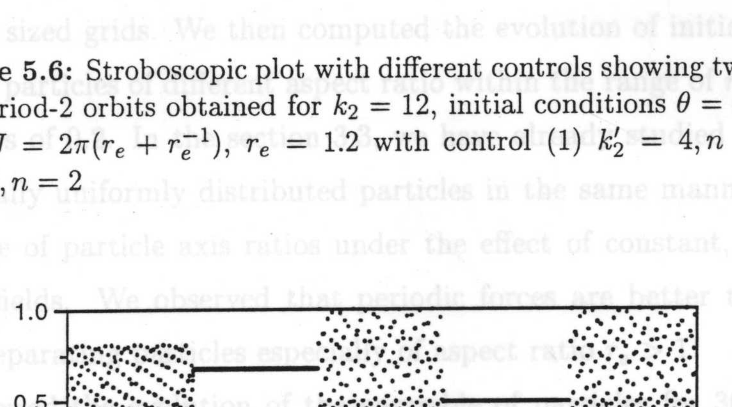

Figure 5.7: Stroboscopic plot showing period-2 and period-3 orbits successively obtained with control applied at every second and third periods respectively for $k_2 = 12$, initial conditions $\theta = \phi = 45°$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $r_e = 1.6, k_2' = 5.0$.

be seen from Tables. 5.3, 5.4 and 5.5.

The observations made in this work suggest the possibility of separating particles more efficiently based on control of chaotic dynamics. We applied a control force (constant force) of magnitude $1 \leq k_2' \leq 5$. For this range of k_2', the chaotic dynamics of particles of all aspect ratio except r_e =0.2, 0.4 and 0.6 could be controlled to a desired orbit. This range of control force seems to be sufficient for efficient separation of particles by shape since particles of these aspect ratios alone can be brought to a desired position. To develop quantitative results based on this observation, we divided the range of possible orientations namely [$-90°$, $90°$] in both θ and ϕ variables into 7 equal intervals resulting in 49 equal sized grids. We then computed the evolution of initially uniformly distributed particles of different aspect ratio within the range of r_e equal to 0.2 to 2 in steps of 0.2. In the section 3.3, we have already studied the evolution of the initially uniformly distributed particles in the same manner within the above range of particle axis ratios under the effect of constant, periodic and zero force fields. We observed that periodic forces are better than constant forces for separating particles especially of aspect ratio $r_e > 1$.

We followed the evolution of the ensemble of particles for 3000 iterations of the stroboscopic plot and deleted the first 1000 values to remove transients in the case of constant and zero forces and in the case of periodic forces with control and without control. In all cases we calculated the number of particles in each grid on every second iteration of the Poincaré section of the evolution equations resulting in a total of 1000 values. We noted the grids in which the total number of particles was greater than or equal to 5 and also noted the number of particles in each grid only if the particle occurred in that grid in more than 100 iterations in all cases. We denote these values as r_e, l_1, l_2 where r_e, l_1, l_2 denote the aspect ratio, total number of occurrences of the grid and total number of particles in the grid on the average respectively and prepared tables for these values. A particle of a given aspect ratio visiting a given grid

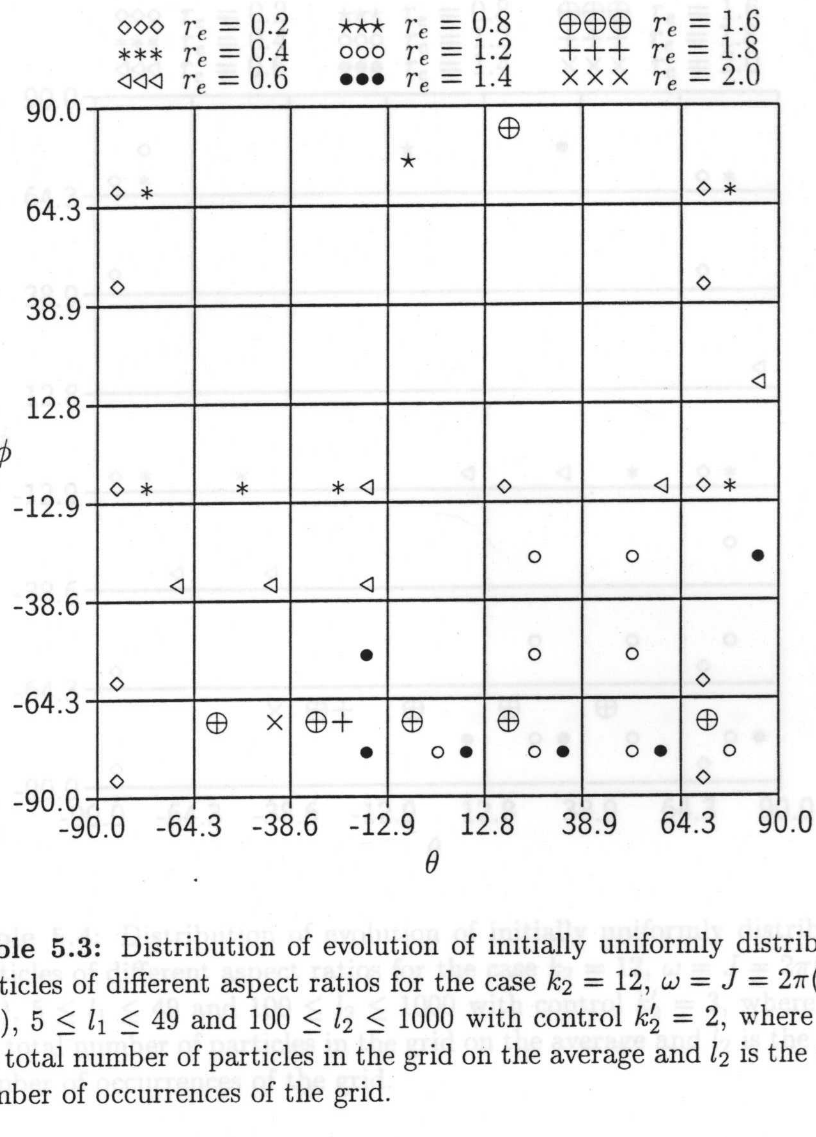

Table 5.3: Distribution of evolution of initially uniformly distributed particles of different aspect ratios for the case $k_2 = 12$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $5 \leq l_1 \leq 49$ and $100 \leq l_2 \leq 1000$ with control $k_2' = 2$, where l_1 is the total number of particles in the grid on the average and l_2 is the total number of occurrences of the grid.

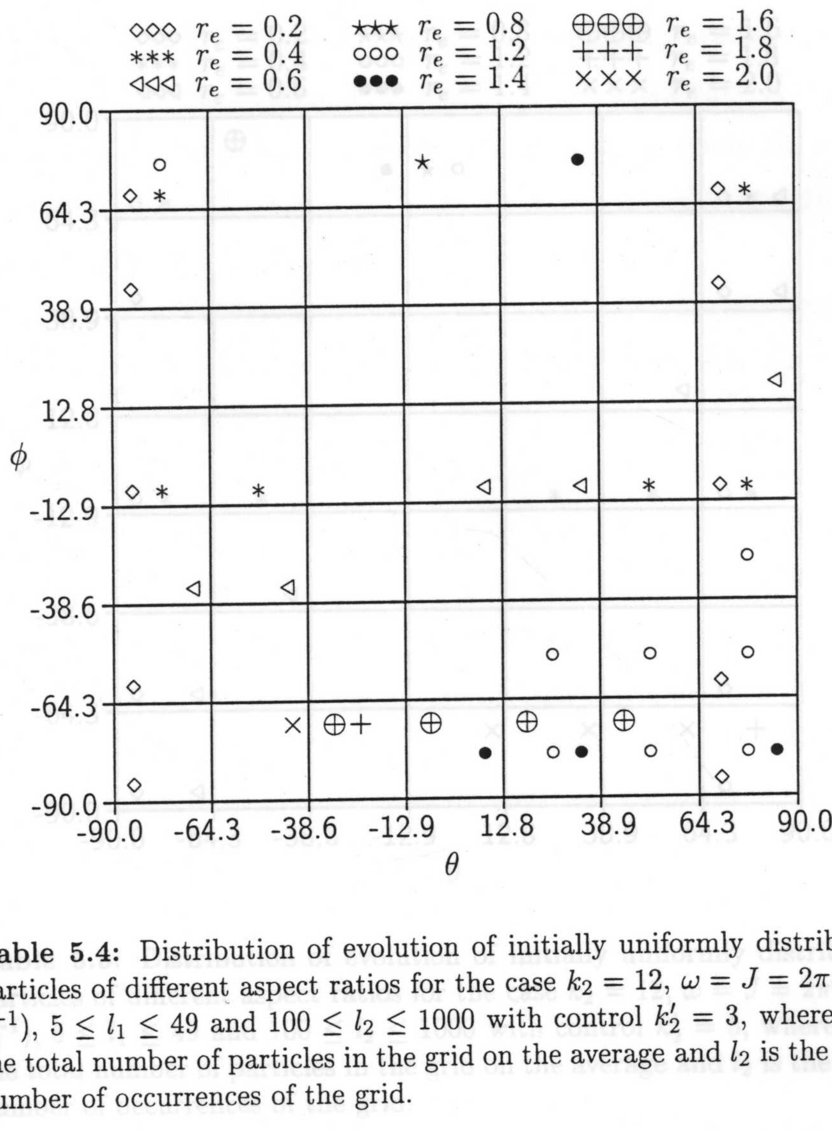

Table 5.4: Distribution of evolution of initially uniformly distributed particles of different aspect ratios for the case $k_2 = 12$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $5 \leq l_1 \leq 49$ and $100 \leq l_2 \leq 1000$ with control $k_2' = 3$, where l_1 is the total number of particles in the grid on the average and l_2 is the total number of occurrences of the grid.

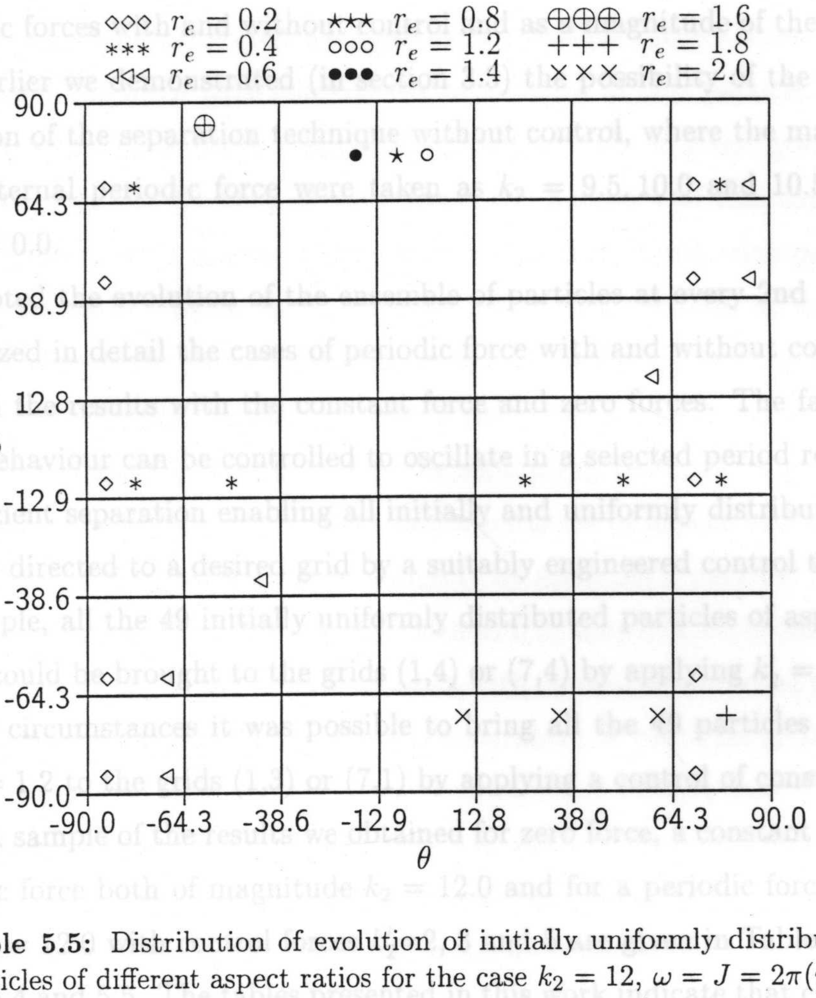

Table 5.5: Distribution of evolution of initially uniformly distributed particles of different aspect ratios for the case $k_2 = 12$, $\omega = J = 2\pi(r_e + r_e^{-1})$, $5 \leq l_1 \leq 49$ and $100 \leq l_2 \leq 1000$ with control $k_2' = 5$, where l_1 is the total number of particles in the grid on the average and l_2 is the total number of occurrences of the grid.

and the absence of particles of other aspect ratios visiting the same grid is more important than the number of visitations of a given grid and the number of particles visiting a grid from the point of view of particle separation. Since our earlier computations indicated the greatest sensitivity of the results to the aspect ratio near $k_2 = 10$, we selected k_2 equal to 12.0 for comparing the effects of periodic forces with and without control and as a magnitude of the constant force. Earlier we demonstrated (in section 3.3) the possibility of the potential application of the separation technique without control, where the magnitudes of the external periodic force were taken as $k_2 = 9.5, 10.0$ and 10.5 keeping $k_1 = k_3 = 0.0$.

We noted the evolution of the ensemble of particles at every 2nd iteration. We analyzed in detail the cases of periodic force with and without control and compared the results with the constant force and zero forces. The fact that a chaotic behaviour can be controlled to oscillate in a selected period resulted in more efficient separation enabling all initially and uniformly distributed particles to be directed to a desired grid by a suitably engineered control technique. For example, all the 49 initially uniformly distributed particles of aspect ratio $r_e = 1.2$ could be brought to the grids (1,4) or (7,4) by applying $k_2' = 4$. Under the same circumstances it was possible to bring all the 49 particles of aspect ratio $r_e = 1.2$ to the grids (1,3) or (7,1) by applying a control of constant force $k_2' = 5$. A sample of the results we obtained for zero force, a constant force and a periodic force both of magnitude $k_2 = 12.0$ and for a periodic force of magnitude $k_2 = 12.0$ with control forces k_2'=2, 3 and 5 are given in Tables 3.1, 5.1, 5.2, 5.3, 5.4 and 5.5. The tables presented in this work indicate that controlling chaotic dynamics is preferable to chaotic behaviour and regular behaviour for efficient separation of particles.

A detailed analysis of all the tables indicates that particles of aspect ratio 0.2 alone can be separated from a mixture containing particles of different aspect ratios ranging within 0.2 to 2.0 by applying a control force of magnitude

k_2'=1, 2, 3, 4 or 5 applied at every second period along with $k_2 = 12$, since particles of aspect ratio 0.2 alone occur in some grids as can be seen from the sample tables. In the case of periodic forces without control and constant and zero forces particles of aspect ratio 0.2 alone also occur along the boundary of the tables. We have suggested in the section 3.3 that particles of this aspect ratio alone could be separated by applying a constant force. However in the case of a periodic force with control, the occurrence of this particle alone are concentrated among a fewer number of grids and visit a given grid a larger number of times. In the case of zero forces the particles of this aspect ratio alone are concentrated among a fewer number of grids along with particles of higher aspect ratio. Thus, for the separation of the particles of this aspect ratio alone a periodic force of magnitude $k_2 = 12.0$ with control is preferable to any other possibility. Similar analysis of the tables shows that it is desirable to apply a periodic force with control for separating particles of aspect ratio 0.4. As can be seen from the sample tables particles of aspect ratio 0.6 alone and 0.8 alone also can separated individually by applying a periodic force with control. In the case of particles of aspect ratio 0.8, they can be brought to one of the extreme grids. Thus, a periodic force with control seems to be preferable to any other case considered for effective separation of particles in the case of aspect ratios $r_e < 1$.

In an earlier analysis made in section 3.3, we have demonstrated that periodic forces are preferable to constant forces and zero forces for separating particles of aspect ratio $r_e > 1$. In the analysis we observed that the occurrence of particles of aspect ratio $r_e > 1$ are spread among a large number of grids. However for particles of aspect ratio greater than or equal to 1.6 individual separation was not possible in the analysis. Further study of the tables prepared in this work indicate that a periodic force with control is once again preferable in such cases. In the case of particles of aspect ratio $r_e > 1.0$ all the initially uniformly distributed particles of the same aspect ratio could be brought to one grid except for $r_e = 1.8$. Hence particles of aspect ratio within the range of r_e ranging from 1.2 to 2.0 except for r_e=1.8 can be easily separated as can be seen from the sample tables prepared. For particles of aspect ratio equal to 1.8 individual separation may not be possible in the cases considered in this work since particles of this aspect ratio appear in combination with particles of lower aspect ratio equal to 0.2 in some cases considered with control. Hence if particles of aspect ratio 0.2 have been separated out as explained earlier, particles of aspect ratio equal to 1.8 can be separated from the mixture. One advantage of periodic forces with control is that all particles of aspect ratio 1.8 can be brought to one grid in combination with the particles of aspect ratio 0.2 as shown in Table 5.5.

Note that the dynamics of periodically forced spheroids of aspect ratios $r_e = 1.8$ and 2.0 is non-chaotic when $k_2 = 12.0$. In this case we lose the flexibility of forcing the particles to oscillate in a desired orbit even though all the particles of these aspect ratios could be brought to a single grid. Hence it may be possible to separate out the particles of aspect ratios $r_e = 1.8$ and 2.0 alone by selecting some other values of k_2 for which the system behaves chaotically. In such cases more efficient separation of individual particles of these aspect ratios may also be possible by a suitable control strategy. This also shows that a chaotic behaviour of the dynamics with or without control is a must for the separation of particles individually.

In conclusion, it has been generally noted that control of chaotic behaviour gives better separation than chaotic and regular behaviour. One of the main features of the method suggested in this work is that all particles of the same aspect ratio can be concentrated in a previously desired grid. A detailed analysis of the problem can suggest suitable designs for separation of particles by aspect ratio to get a well characterized suspension of particles. As pointed out in the section 3.3, a possible design for this separation of particles from a mixture of particles of different aspect ratios based on the differences in the orientation of the particle may consist of a base plate having grooves along different orientations so that when the particles are oriented in such directions, they settle in a particular groove and can be separated out at every integral multiple of the fundamental period.

Thus, even in the rather simple application considered in this work, control of chaos leads to greater efficiency. This suggests that the possibility of chaos control should be important in many of the applications mentioned in the introduction. The novel control of chaos technique suggested in this section has been demonstrated to be effective even in the relatively complex problem considered here. An additional feature of the control of chaos technique suggested is that the control is effected very rapidly and the behaviour of the concerned system can be switched from one desired period to another desired period very rapidly. One of the interesting results noted is the possibility to stabilize periodic orbits of period appreciably greater than by the Ott *et al.* method (Ott *et al.* 1990). This suggests that this control of chaos technique may be applied to other chaotic systems very effectively.

One paper resulting from this work is in press in the journal **Sadhana**, published by Indian Academy of Sciences.

We compared the new technique in three dynamical systems and found some advantages over two other techniques. A detailed comparative analysis of the new control of chaos algorithm on some physically realizable model systems will be presented in the next chapter.